

The Level Splitting Distribution in Chaos-assisted Tunneling

François Leyvraz^{1*} and Denis Ullmo^{1,2}

¹*Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France*

²*AT&T Bell Laboratories 1D-265, 600 Mountain Avenue, Murray Hill, New Jersey
07974-0636, USA*

(February 5, 2008)

Abstract

A compound tunneling mechanism from one integrable region to another mediated by a delocalized state in an intermediate chaotic region of phase space was recently introduced to explain peculiar features of tunneling in certain two-dimensional systems. This mechanism is known as chaos-assisted tunneling. We study its consequences for the distribution of the level splittings and obtain a general analytical form for this distribution under the assumption that chaos assisted tunneling is the only operative mechanism. We have checked that the analytical form we obtain agrees with splitting distributions calculated numerically for a model system in which chaos-assisted tunneling is known to be the dominant mechanism. The distribution depends on two parameters: The first gives the scale of the splittings and is related to the magnitude of the classically forbidden processes, the second gives a measure of the efficiency of possible barriers to classical transport which may exist in the chaotic region. If these are weak, this latter parameter is irrelevant; otherwise it sets an energy scale at which the splitting distribution crosses over from one type of behavior to another. The detailed form of the crossover is also obtained and found to be in good agreement with numerical results for models for chaos-assisted tunneling.

Typeset using REVTeX

*Permanent address: Laboratorio de Cuernavaca, Instituto de Física, Universidad Nacional Autónoma de México, Apdo. postal 20-3643, 01000 México D.F., Mexico

I. INTRODUCTION

Under the denomination of “quantum chaos”, a large body of theoretical and experimental work has been, and continues to be, devoted to the study of the specific features of a quantum system which can be traced back to the degree of chaoticity of the underlying classical dynamics [1,2,3]. Among them, questions concerning tunneling effects receive now an increasing degree of attention. Indeed, although they are often considered as purely quantum, since they correspond to classically forbidden events, it appears amply clear now that tunneling processes are strongly affected by the nature of the underlying classical dynamics [4,5,6,7,8,9].

This is made most explicit when considering time-dependent systems as well as systems in more than one dimension, for which, in contradistinction to the one- d conservative problems usually considered in basic quantum mechanics textbooks, energy conservation does not constrain the motion to be integrable. A first consequence of having a richer dynamics is that quasi-degeneracies analogous to those found in the standard symmetric double well problem can be observed in systems possessing a discrete symmetry, independently of whether a potential barrier actually exists or not. Indeed, unless the dynamics is entirely ergodic, some classical trajectories are trapped in d -dimensional invariant manifolds (the invariant tori) inside the $2d$ -dimensional phase space. If there is a discrete symmetry, say parity or time reversal, in a given system, then any tori which are not themselves invariant under this symmetry operation will come in symmetrical pairs. Semiclassical EBK quantization can be applied to these symmetrical tori, and, since they are entirely identical, this approximation yields degenerate energy levels. Note that this effect is distinct from ordinary symmetry induced eigenvalue degeneracy. Indeed, the effect is only semiclassically correct and tunneling effects lift this degeneracy. This kind of tunneling has been dubbed “dynamical tunneling ” by Davis and Heller [10].

Rephrasing this in terms of wave functions dynamics makes it clearer why the term “tunneling” is proper in spite of the absence of a barrier. Indeed, it is possible to construct what Arnold has termed *quasi-modes* [11] whenever there exist invariant tori fulfilling the EBK quantization conditions [12]. These are wave-functions semiclassically constructed on a single torus, and *which fulfill the Schrödinger equation up to any order in \hbar* . Here, similarly to what happens when a potential barrier actually exists (e.g. a one- d conservative system), the actual eigenstates are not approximated by a single quasi-mode, but rather by a linear combination of quasi-modes constructed on symmetry related tori. Therefore, if one allows a quasi-mode constructed on one of the tori, to evolve for a very long time, it will eventually evolve into its symmetric partner, whereas classically trajectories remain indefinitely trapped on one torus.

Our interest in higher dimensional systems lies however mainly in the possibility of considering dynamics of a different nature. Even for integrable systems, no general theory of tunneling in multidimensional systems is presently available. However some theoretical and numerical studies [13,14] clearly demonstrate that the tunneling mechanism in this case is rather similar to what is observed for one- d systems. In particular the splitting between two quasidegenerate doublets has a smooth exponential dependence in \hbar .

If on the contrary the dynamics is mixed, (as will generically be the case for low dimensional Hamiltonian systems), and one is interested in the tunneling between two symmetry

related invariant tori separated by a significant chaotic region, new behavior has been observed, which is quite different from what one is used to in the integrable case. From a now growing body of numerical work, either on one-dimensional time dependent systems [4,5,6], or on two dimensional conservative systems [7,8], it has become clear that the presence of chaos is associated with certain qualitative features, namely: (i) great enhancement of the average splitting, (ii) extreme sensitivity to the variation of an external parameter, (iii) strong dependence of the tunneling properties on what is going on in the chaotic region separating the two tunneling tori. A particularly striking evidence of this last point is given in [8], where it is observed that reducing drastically the classical transport in the chaotic sea from the neighborhood of one torus to the one of its symmetric partners noticeably reduces the tunneling rates.

In [9], it has been demonstrated that a natural interpretation of this unusual tunneling behavior is obtained if one consider that instead of the “direct” tunneling mechanism characteristic of the integrable regime, a new tunneling mechanism takes over, in which the chaotic region plays a predominant role. In “direct” tunneling, splittings are caused by the overlap of the semiclassical functions constructed on one torus of the symmetrical pair via the EBK scheme (the “quasi-modes”). As discussed above, these quasi-modes are not eigenstates, but fulfill the Schrödinger equation up to arbitrary order in \hbar . Nevertheless, they can be connected to other states through “tunneling” matrix elements which are exponentially small in \hbar . (See section II.A of [9] for an explicit example.) In the “chaos-assisted” regime on the contrary, the picture is that the particle first tunnels from the integrable region to the chaotic sea and from there again to the symmetrical region. More precisely, this means that the tunneling is dominated by the matrix elements between the quasi-modes and states semiclassically localized in the chaotic region, rather than by the matrix element connecting directly the two quasi-modes. This process involves one (or many) intermediate state and two tunneling processes. The reason why it may nevertheless dominate is that, since the chaotic region is much closer in phase space than the symmetrical integrable region, the tunneling amplitudes are expected to be usually much larger. Once in the chaotic region, there is nothing to stop the particle from reaching the symmetrical integrable region. In this process, not only do the two semiclassical states play a role, but the various delocalized chaotic eigenstates which might couple to them also become relevant. For this reason, the tunneling amplitudes have a remarkable dependence on \hbar : first, they decay exponentially fast with \hbar , reflecting the smooth variation of the tunneling amplitude from the integrable region to the chaotic sea. Second, however, and superposed upon this smooth variation, there is an extremely irregular fluctuation of the splittings, due to the violent variation in the strength of the coupling to the nearby-lying chaotic states. This depends very sensitively on the smallness of the energy denominator, i.e. on whether or not a chaotic level lies close to the tunneling doublet. In fact, these fluctuations can be so strong as to make any realistic assessment of the \hbar dependence of the tunneling amplitude impossible. Note however that this picture may be oversimplified: it was observed on the kicked Harper model by Roncaglia et al. [15], that direct tunneling can be dominant even in the presence of a significant chaotic region.

That the mechanism described above is actually the one taking place for tunneling in the presence of chaos cannot, at the present time, be derived from the basic quantum mechanical law of evolution. The numerical evidence as well as a far more careful and detailed discussion

of the process are given in [9]. It is worth stressing, however, that one of the most important implications of the chaos-assisted mechanism described above is that it allows for a modeling of the splitting distribution in terms of ensembles of random matrices, as discussed in more detail in [9]. Therefore not only a qualitative interpretation of the numerically observed tunneling behavior is obtained, but also *quantitative* theoretical predictions can be made, and compared to numerically obtained data with a surprisingly good accuracy.

The purpose of this paper being the study of the resulting matrix ensembles, let us be more specific on what is understood by “splitting distribution”, and how the theoretical predictions have been obtained in [9]. As mentioned above, a characteristic feature of tunneling in the presence of chaos is the extreme sensitivity of such quantities as splittings, to the variation of an external parameter. Within the “chaos-assisted” interpretation of the tunneling mechanism, this is quite natural since the splitting of a given doublet may vary by orders of magnitudes depending on whether a chaotic state is close to the tunneling doublet or not. Therefore, very small changes of external parameters, which leave the classical dynamics almost unaltered, may drastically change the splitting by shifting chaotic levels a distance of a few mean spacings. Therefore, in any experimental setting the statistical behavior of the splittings is likely to be of great relevance, even if one focuses on one single well-defined doublet. In this case, the physically relevant quantity will be the distribution of splittings for an ensemble obtained by varying an external parameter over a range which is negligible on the classical scale, but still large on the quantum scale (i.e. the tunneling doublet crosses a lot of chaotic levels). By analogy with the random matrix ensembles describing the spectral fluctuations of classically chaotic systems¹ (see e.g. [16,17]), or to their generalization introduced to describe partly chaotic systems (see section 5 of [7]), one can build random matrix ensembles modeling the tunneling distributions corresponding to a given phase space structure.

We shall not repeat here in detail the prescription given in [9] for constructing the random matrix ensemble relevant to a given classical configuration. (A few examples of such ensembles are given below.) The following points, however, should be stressed: First, we are not considering fully ergodic systems (otherwise there would be no invariant tori); hence the chaotic part of phase space cannot a priori be considered as structureless. A whole set of partial barriers should quite generally be present, preventing the motion in the chaotic region from being completely random. In order to quantify the efficiency of these partial barriers, additional time scales must be introduced apart from the mean Lyapunov exponent. As a consequence, the random matrix ensembles associated to the chaotic part of the phase space cannot be taken as structureless either, as demonstrated in [7]. One has to introduce “transition ensembles” entirely specified by a set of “transport parameters” $\Lambda_1, \Lambda_2, \dots$. These transport parameters are fixed by the classical dynamics, and are therefore not adjustable model parameters. Therefore the parameters which determine the tunneling distribution are, in the chaos-assisted regime, of two kinds:

—(i) the variance v_t^2 of the tunneling matrix elements. It describes the classically forbidden

¹These were originally introduced in Nuclear Physics without any intention to discuss the nature of the underlying classical dynamics

part of the tunneling process. There exists, at the present time, no theoretical way to evaluate it.

—(ii) the set of “transport parameters” $\Lambda_1, \Lambda_2, \dots$, which are fixed by the classical dynamics inside the chaotic region.

A second thing we would like to stress is that it is not necessary to solve analytically the thus constructed random matrix ensembles to obtain a “theoretical prediction” for the splitting distributions. Indeed the latter are entirely specified by the random matrix ensemble, and can be obtained concretely by performing a rather straightforward Monte Carlo calculation: i.e. taking at random a large number of matrices with the distribution specified by the ensemble, diagonalizing them numerically to obtain the splitting, and construct in this way a histogram of the splitting distribution. This was the procedure used in [9] to compare splitting distributions of doublets of a system of coupled quartic oscillators with those predicted by the proper random matrix ensemble.

The obvious drawback of using Monte Carlo simulations to produce the matrix ensembles prediction is that it sheds little light on the characteristic features of the distribution. This is made even more important here because, although all parameters but the tunneling amplitude v_t can in principle be computed by studying the classical motion in the chaotic region, their practical calculation requires a great deal of effort in the simplest situations, and could turn out to be impossible for sufficiently complicated classical structures. Moreover, in experimental realizations, the Hamiltonian governing the dynamics may not be known in enough detail to allow fixing the parameters of the ensemble with sufficient confidence.

It is therefore worthwhile to gain some further understanding of the splitting distribution determined by the ensembles of random matrices constructed in [9], and to obtain explicit expressions for these distributions. As we shall see, they can in fact be expressed in rather simple form. Moreover, the resulting distributions have some very specific general feature, which can be used as *the fingerprint of chaos assisted tunneling* even when the precise structure of the chaotic motion is unknown. Because of the relative complexity of the derivation, we have chosen to organize this paper as follows. In section II, we shall give the final result, without any justifications, and show how well our analytic findings compare with actual splitting distributions, obtained in [9] for a system of coupled quartic oscillators. The remaining of the paper will be devoted to the derivation of this result. Section III will deal with the simpler case where the chaotic phase space can be taken as structureless. In section IV we shall derive the splitting distributions in the case where effective partial barriers are present, and discuss in more details the hypothesis and approximation used in the derivation. Section V will be devoted to some concluding remarks.

II. THE SPLITTING DISTRIBUTION

In the following we introduce some notation. We are interested in a system possessing a discrete symmetry P , and for which tunneling takes place between two quasimodes Ψ_R^1 and Ψ_R^2 constructed on symmetrical invariant tori \mathcal{T}_1 and $\mathcal{T}_2 = P(\mathcal{T}_1)$. The eigenstates belong to a given symmetry class $+$ or $-$ depending on whether they are symmetric or antisymmetric under the action of P . We note Ψ_R^+ and Ψ_R^- the symmetric and antisymmetric combinations of the quasimodes Ψ_R^1 and Ψ_R^2 . If one neglects the direct coupling between the quasimodes,

Ψ_R^+ and Ψ_R^- have the same mean energy E_R . The “chaos-assisted” mechanism proposed in [8] assumes that the tunneling from \mathcal{T}_1 to \mathcal{T}_2 originates from the (exponentially small) coupling between Ψ_R^+ (resp. Ψ_R^-) and chaotic states of same symmetry $|n, +\rangle$ ($n = 1, 2, \dots$) (resp. $|n, -\rangle$) semiclassically localized in the chaotic region surrounding the islands of stability containing \mathcal{T}_1 and \mathcal{T}_2 . Therefore, in a basis where the chaotic part of the Hamiltonian is diagonal, the $+$ and $-$ sectors appear respectively as

$$H^+ = \begin{pmatrix} E_r & v_1^+ & v_2^+ & \cdot & \cdot & \cdot \\ v_1^+ & E_1^+ & 0 & 0 & & \\ v_2^+ & 0 & E_2^+ & 0 & & \\ \cdot & 0 & 0 & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \end{pmatrix} ; \quad H^- = \begin{pmatrix} E_r & v_1^- & v_2^- & \cdot & \cdot & \cdot \\ v_1^- & E_1^- & 0 & 0 & & \\ v_2^- & 0 & E_2^- & 0 & & \\ \cdot & 0 & 0 & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \end{pmatrix}. \quad (2.1)$$

After diagonalization of H^+ and H^- , the regular levels will be shifted respectively from an amount δ^+ and δ^- , giving the splitting

$$\delta = |\delta^+ - \delta^-|. \quad (2.2)$$

Eq. (2.1) merely summarizes the semiclassical picture one has of the chaos assisted tunneling, but no random matrix modeling has been introduced yet. This latter is obtained by assuming that the statistical properties of the tunneling is correctly reproduced if one model H^+ and H^- by some ensemble of matrices with a specified distribution.

As discussed in detail in [9], the natural choice for the tunneling matrix elements v_n^\pm describing the classically forbidden process is to take them as independent Gaussian variables with the same variance v_t^2 , and, in the absence of any partial barrier, to use for the chaotic levels $E_1^+, E_2^+, \dots, E_1^-, E_2^-, \dots$ the classical ensembles of Wigner and Dyson which are known to model properly the spectral statistics of completely chaotic systems [18]. If time reversal invariance symmetry holds, as we shall assume in the following, this means that one should take E_1^+, E_2^+, \dots and E_1^-, E_2^-, \dots as two independent sequences, with a distribution given by the Gaussian Orthogonal Ensemble (GOE). Symbolically this ensemble is denoted by

$$H^+ = \begin{pmatrix} E_R & \{v\} \\ \{v\} & (GOE)^+ \end{pmatrix} ; \quad H^- = \begin{pmatrix} E_R & \{v\} \\ \{v\} & (GOE)^- \end{pmatrix}, \quad (2.3)$$

where the subscripts $+$ and $-$ emphasize the independent nature of the distribution. In that case, we shall see in section III that the splitting distribution $p(\delta)$ is merely a truncated Cauchy law

$$p(\delta) = \frac{4v_t}{\delta^2 + 4\pi v_t^2} \quad (\delta < v_t), \quad (2.4)$$

$$p(\delta) = 0 \quad (\delta > v_t).$$

As demonstrated in [7], such a simple statistical modeling of the chaotic states does not apply any more when structures, such as partial barriers, prevent classical trajectories to flow freely from one part of the chaotic phase space to another. In such classical configurations (which are presumably generic in systems where chaos and regularity coexist), transition

ensembles have to be introduced to obtain a correct statistical description of the chaotic states. Compared to the case where no barriers are present, the distribution of chaotic states will be modified in two ways. First, each of the parity sequences, taken separately, will usually not be distributed as a GOE anymore. However, as stressed in [9], and as will be made extremely clear in section III, this has a negligible influence on the splitting distribution. More important is that such barriers may induce strong correlations between the two parity sequences of chaotic states. To fix ideas, let us consider a simple example for which a strong partial barrier separates the chaotic sea into two parts \mathcal{R}_1 and \mathcal{R}_2 which are symmetric images one of each other under P . In that case, the relevant matrix ensemble can be symbolically written as [9]

$$H^\pm = \begin{pmatrix} E_R & \{v\} \\ \{v\} & (GOE)_S \pm (GOE)_A(\Lambda) \end{pmatrix}, \quad (2.5)$$

where the variance of the matrix elements of $(GOE)_S$ is chosen such that it has (in the neighborhood of E_R) the same mean spacing D as the chaotic states, and the variance σ^2 of the matrix elements of $(GOE)_A(\Lambda)$ is fixed by the transport parameter Λ through

$$\frac{\sigma^2}{D^2} \equiv \Lambda. \quad (2.6)$$

The transport parameter is in turn semiclassically related to the classical flux Φ crossing the partial barriers by [9]

$$\Lambda = \frac{1}{4\pi^2} \frac{g\Phi}{(2\pi\hbar)^{d-1} f_1 f_2}, \quad (2.7)$$

where $g = 1/2$ is the proportion of states in the corresponding symmetry class, $f_1 = f_2 = 1/2$ the relative phase space volume of region 1 and 2 and d the number of freedoms.

For very ineffective barriers, Λ will be much larger than 1. $(GOE)_S + (GOE)_A(\Lambda)$ and $(GOE)_S - (GOE)_A(\Lambda)$ will then be two essentially independent ensembles and one will recover in that case the truncated Cauchy law Eq. (2.4) for the splitting distribution. At the opposite extreme, a perfect barrier would give $\Lambda = 0$ (except for classically forbidden processes), and the $+$ and $-$ spectra will be strictly identical. In particular so will be the displacements δ^+ and δ^- , giving a null splitting. For small, but finite Λ , a typical level E_n^- will be usually found close to its symmetric analog E_n^+ , though slightly shifted. In this case, although the displacements δ^+ and δ^- are still distributed as Cauchy, they are strongly correlated. This in turn noticeably affects the splitting δ .

Here it may be useful to give a qualitative description of the meaning of Λ . It can be described as the ratio t_H/t_c of two timescales, one of which is purely classical, whereas the other is essentially quantum mechanical: t_c is the time necessary to cross the barrier, that is, the typical time that a classical trajectory needs in order to go from one part of the chaotic sea to the other. The second time scale t_H is the Heisenberg time \hbar/D , where D is the average level spacing in the chaotic sea. Eventually, of course, one expects Λ to go to infinity, as the Heisenberg time becomes classically infinite in the semiclassical limit. Nevertheless there may well be a very large intermediate region, for which Λ takes very small values. In such cases, there exists a classical time scale comparable or larger than

the Heisenberg time. This time scale is related to the time necessary to explore all of the available phase space. In this respect the situation is quite reminiscent of what happens in localization. The difference, of course, is that we only have a small number of weakly connected phase space regions and the long classical time has nothing to do with diffusion.

Usually, the ensemble describing the classical structure of a system will be more complicated than the simple one given in Eq. (2.5). Indeed, it is probable that transport from one regular island to its symmetric counterpart is affected through the joint effect of a whole set of moderately effective barriers rather than by the strong action of a single one. Thus one will have to consider much more structured ensembles, with a transport parameter Λ_n associated to each barrier. Under these circumstances, there is a possibility that localization effects begin to play a non-negligible role. Should localization become as effective in limiting the quantum transport as the classically forbidden processes, our treatment would become irrelevant. On the other hand we shall see in section IV that when this is not the case a noticeable simplification of the problem occurs because all the information encoded in the transition ensemble (that is, essentially, the Λ_n 's) can be summarized in a single parameter α , which is a weighted average of the variance of the $(E_n^+ - E_n^-)$.

The splitting distribution $p(\delta)$ therefore depends on three parameters: $(v_t)^2$, the variance of the tunneling matrix elements, α^2 which measures the degree of correlation between the odd and even levels, and D , the mean spacing of the chaotic levels to which the Ψ_R^\pm are connected. If one considers effective barriers, α is smaller than D . Moreover, v_t being related to classically forbidden processes is usually extremely small, and in particular much smaller than α . We shall therefore assume below $v_t \ll \alpha < D$. Then, the main result of this paper is that, for this parameter range, the splitting distribution is given by:

- for $v_t < \delta$

$$p(\delta) = 0 , \quad (2.8)$$

- for $v_t^2/\alpha < \delta < v_t$

$$p(\delta) = \frac{4v_t}{\delta^2 + 4\pi v_t^2} \quad (\text{Cauchy}) , \quad (2.9)$$

- for $\delta < v_t^2/\alpha$

$$p(\delta) = 2\mu^{-1}G\left(\frac{\delta}{\mu}\right) , \quad (2.10)$$

where the function G is the inverse Fourier transform of $\exp(-\sqrt{|q|})$, namely

$$G(x) \equiv \frac{1}{2\pi} \int \exp(iqx) \exp(-\sqrt{q}) dq , \quad (2.11)$$

$$\text{and} \quad \mu = \frac{\sqrt{32}\Gamma^2(3/4)}{\pi} \frac{\alpha v_t^2}{D^2} . \quad (2.12)$$

As expected (see section IV.B. of [9]), only the smaller splittings are affected by the transport limitation, the distribution for larger splittings being unaffected. The asymptotic behavior of the $\mu^{-1}G(\delta/\mu)$ is given by

$$p(0) = 2\mu^{-1}G(0) = \frac{1}{\sqrt{2}\Gamma^2(3/4)} \frac{D^2}{\alpha v_t^2} \quad \text{in } \delta = 0 \quad (2.13)$$

$$\mu^{-1}G(\delta/\mu) \simeq \frac{\Gamma(3/4)}{2^{1/4}\pi} \frac{\sqrt{\alpha}v_t}{D} \frac{1}{\delta^{3/2}} \quad \text{for } \delta \gg \alpha v_t^2/D^2 \quad (2.14)$$

Therefore, for small enough α/D and v_t/D , the distribution $p(\delta)$ will, in a log-log plot, essentially consist of three straight lines: (i) a horizontal one (at $p(0)$, as given by Eq. (2.13)), for $0 < \delta < \alpha v_t^2/D^2$. (ii) a line of slope $(-3/2)$ in the range $\alpha v_t^2/D^2 < \delta < v_t^2/\alpha$. (iii) a slope (-2) characteristic of the Cauchy distribution for the range $v_t^2/\alpha < \delta < v_t$, after which the distribution brutally falls to zero.

Here let us digress shortly to give an intuitive picture of what is happening. Let us first assume that there are no barriers. Then chaos-assisted tunneling is essentially a compound process involving the two symmetrical states and those chaotic states which lie nearby. Fast tunneling (large splittings) occur only if at least one of these states actually lies very near to the quasi-degenerate tunneling state. This then yields a tunneling process mediated by one single delocalized chaotic state. This process has a characteristic δ^{-2} distribution, as we shall show later. Here it is essential to realize that the chaotic state involved, since it has a well-defined symmetry, will always directly couple from one torus to its symmetrical partner. On the other hand, if we have an efficient barrier, the chaotic states also come in quasi degenerate doublets of opposite parity and of width α . Therefore, moderately fast processes will be mediated by such a doublet rather than by a single state. Again, we shall show that this leads to a universal behavior of $\delta^{-3/2}$ as long as the doublet is identifiable as such, that is, as long as the two energy denominators contribute roughly equally. However, for very fast processes, the tunneling will again be mediated by a single state, namely the one nearest to the tunneling doublet, and the δ^{-2} behavior is again obtained. The details are given by the above formulae, which also show a considerable amount of information for intermediate cases which cannot be derived in such a simple fashion.

Before going to the calculation of the above distribution, let us see how it compares to actual distributions of splittings obtained numerically for a system of two coupled quartic oscillators governed by an Hamiltonian of the form

$$H(\mathbf{p}, \mathbf{q}) = \frac{p_1^2 + p^2}{2m} + a(q_1^4/b + bq_2^4 + 2\lambda q_1^2 q_2^2) \quad (2.15)$$

Except for their presentation (we use here a log-log plot instead of a linear versus log binning), the data used in Fig. 1 are exactly the same as those used in Fig. 13 of Ref. [9], to which we refer the reader for more precise information on the system investigated. Here, we shall only say that each set of data has been obtained by numerically calculating the splittings between regular states constructed on a given identified pair of symmetrical invariant tori, for various values of the coupling λ .² The range of variation of λ is small on the classical scale

²In practice, to increase the statistical significance of the distribution, data coming from close

(the classical dynamics remains essentially the same), but sufficiently large on the quantum scale that a good statistical significance is reached.

In Fig. 1 we display the comparison between the quartic oscillators data and the predicted form of the distribution Eqs. (2.8)-(2.10) for two splitting distributions associated to two different pairs of symmetric invariant tori. The agreement is extremely good, especially if one considers that the distribution extends over more than six decades. Here a remark is in order. The parameters α and v_t used for the analytical curves in Fig. 1 are actually tunable parameters. This was already the case for v_t for the ensemble introduced in [9], since there is not yet any semiclassical theory allowing for the calculation of the matrix elements associated to such classically forbidden processes. Here however, one has another tunable parameter α . In principle, this parameter is fixed once the random matrix ensemble describing the statistical properties of the chaotic level is known, which is the case for this particular system. In practice however, there is usually no way to relate α analytically to, say, the set of transport parameters Λ_n 's. Therefore α eventually plays the role of a tunable parameter. It should be borne in mind however that α and v_t only fix the scale of the distribution, and in particular the place of the crossover from Cauchy to the G -like behavior, but not its shape. Therefore, despite the presence of two tunable parameters, the fact that the splitting distribution in Fig. 1 actually follows the prediction of Eqs. (2.8)-(2.10) is a very stringent test of the relevance of the whole “chaos-assisted” picture. After this attempt to put the results in perspective, we turn to their derivation.

III. THE CASE WITHOUT BARRIERS

We now want to get down to computing the distribution of the splittings in the case in which no barriers are present. Under these circumstances, it is sufficient to compute the distribution of δ_+ , and hence δ_- , since these splittings are statistically independent.

To start with, one should note that, when considering the matrices H^+ and H^- in Eq. (2.1), the matrix elements v_i^\pm are associated with classically forbidden processes, and are thus extremely small. Therefore, one can compute the displacement δ^\pm using a first order perturbation result. One has however here to take special care with the rare, but important, case, where a chaotic level come extremely close to E_R . This can be done using the exact two by two diagonalization result for each chaotic eigenstate, and adding up the contributions. This gives

$$\delta^\pm = \frac{1}{2} \sum_{i=1}^N (E_R - E_i^\pm) \left(1 - \sqrt{1 + \left(\frac{2v_i^\pm}{E_R - E_i^\pm} \right)^2} \right) \quad (3.1)$$

(to be understood in the $N \rightarrow \infty$ limit). In the absence of any values of E_i^\pm too close to E_R , the above equation is equivalent to the usual perturbative result,

lying tori have been merged.

$$\delta^\pm \approx \sum_{i=1}^N \frac{|v_i|^2}{E_R - E_i^\pm}, \quad (3.2)$$

but the full expression Eq. (3.1) has to be used to regularize it whenever any of the $(E^\pm - E_R)$ become of the order of v_t .

Although we shall give below a more detailed discussion of that point, the basic way we are going to use Eq. (3.1) is that the regularized form of $\delta^\pm(E_i^\pm, v_i^\pm)$ prevents any splitting from being significantly larger than v_t , and that for δ^\pm smaller than v_t , Eq. (3.2) can be used safely. To clarify the discussion, we shall for the moment replace the $(v_i^\pm)^2$ by their average value v_t^2 , and justify below why this does not change the result. Without loss of generality we also set E_R equal to zero (E_R is not correlated to the chaotic spectrum, so it can be used as the origin of the energies). We shall in addition consider the normalized regular level shift x and energy level e_1, e_2, \dots (we drop the superscript $+$ or $-$ for the normalized quantities)

$$x = \frac{\delta^\pm D}{v_t^2} \quad ; \quad e_i = \frac{E_i^\pm}{D}. \quad (3.3)$$

With these manipulations, $p(x)$ is in principle obtained for $x < D/v_t$ (i.e. $\delta^\pm < v_t$) as the integral

$$p(x) = \int \delta\left(x - \sum_{i=1}^N \frac{1}{e_i}\right) P(e_1, e_2, \dots, e_N) de_1 de_2 \dots de_N, \quad (3.4)$$

where $P(e_1, \dots, e_N)$ is the joint probability of a GOE spectra with mean density equal to one in the center of the semicircle. It appears however that, the correlations of the chaotic states have no influence on $p(x)$, because the physics here is determined by the singular nature of the energy denominator which is not expected to be very sensitive to many-particle correlations.

To demonstrate this point, let us consider for instance the integral Eq. (3.4) except that we take for the chaotic states a Poisson distribution, i.e. that we neglect any correlations between them. In that case, introducing $\xi_i = Ne_i$, one can write

$$x = \frac{1}{N} \sum_{i=1}^N \xi_i^{-1}, \quad (3.5)$$

that is the random variable x is the average of the ξ_i^{-1} , where the ξ_i 's are independent variables with density of probability one at the origin. Would the distribution of $1/\xi_i$ admit a second moment (i.e. $\langle \xi^{-2} \rangle < \infty$), the usual central limit theorem would yield a Gaussian distribution for $p(x)$. Here, however, the situation is quite different since this variance actually diverges. The distribution $p_0(y)$ of the $y_i = \xi_i^{-1}$ behaves as y^{-2} for $y \gg 1$, whatever the initial distribution of the ξ_i , as long as that distribution is equal to one for ξ equal to zero³

³Therefore, the result will not be affected by a secular change of the mean density of states away from the origin.

From this follows through standard probabilistic arguments [19] that x has a non-singular limiting distribution, namely the Cauchy law

$$p(x) = \frac{1}{\pi^2 + x^2} . \quad (3.6)$$

Informally, this result can be obtained as follows. If the chaotic states are distributed independently, Eq. (3.4) reads

$$p(x) = \int_{-\infty}^{\infty} \prod_{i=1}^N dy_i p_0(y_i) \delta \left(x - \frac{1}{N} \sum_{i=1}^N y_i \right) . \quad (3.7)$$

If we now take the Fourier transform of Eq. (3.7), the result factorizes and one obtains

$$\hat{p}(q) \equiv \int_{-\infty}^{\infty} p(x) e^{iqx} dx = \hat{p}_0(q/N)^N , \quad (3.8)$$

where $\hat{p}_0(q)$ is the Fourier transform of $p_0(y)$. The large- y behavior of the latter leads to a singularity of $\hat{p}_0(q)$, which, by reason of the symmetry of $p_0(y)$, must be located at the origin. Further, this singularity is of the type of a discontinuous derivative. For any symmetric function $f(y)$ with the same large y behavior as $p_0(y)$, $f(y) - p_0(y)$ decreases more rapidly than y^{-2} , which implies that $\hat{f}(q) - \hat{p}_0(q)$ has a continuous derivative. The jump in $q = 0$ of the derivative of $\hat{p}_0(q)$ must therefore be the same as for $f(y) = (1 + y^2)^{-1}$, the Fourier transform of which is $\exp(-\pi|q|)$. Noting moreover that, because of the normalization, $\hat{p}_0(0) = 1$, one has

$$\hat{p}_0(q) = 1 - \pi|q| + o(q) \quad (q \ll 1) , \quad (3.9)$$

and therefore

$$\hat{p}(q) = \lim_{N \rightarrow \infty} \left(1 - \frac{\pi|q|}{N} \right)^N = \exp(-\pi|q|), \quad (3.10)$$

from which the result follows immediately by inverse Fourier transformation.

At the opposite extreme, one can consider the most rigid spectrum, and see what happens if the chaotic states are distributed as a picket fence. In that case, $p(x)$ can be written as

$$p(x) = \int_{-1/2}^{+1/2} de \delta \left(x - \sum_{n=-\infty}^{+\infty} \frac{-1}{n + e} \right) \quad (3.11)$$

which, using the equality [20]

$$\cotg(\pi x) = \frac{1}{x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2}$$

readily gives

$$p(x) = \int_{-1/2}^{+1/2} dE \delta(x - \pi \cotg(\pi E)) = \frac{1}{x^2 + \pi^2} , \quad (3.12)$$

that is the very same Cauchy distribution as for the Poissonian case. There is no doubt that if the two extremes of totally uncorrelated and completely correlated spectra give the same result, the correlation between chaotic states play little or no role. Therefore, as demonstrated in Fig. 2, the splittings are also Cauchy-distributed when the chaotic states are GOE distributed. Indeed, this result can actually be shown using supersymmetric techniques [21] and also turns out to follow from results on S -matrix ensembles for the one-channel case [22] under quite general conditions for both the v_i and the energies E_i . In fact, it turns out that the sums involved in computing the K -matrix in a one channel system are exactly of the type we are interested in and their distribution can be found exactly under the assumption that the ensemble is ergodic and analytic in the energy. For details see [22] and [23].

Before turning to the more difficult case of problems where transport limitations play a role in the tunneling mechanism, let us come back to a couple of points not treated in the above discussion. Since we have seen that correlations between chaotic states are of little importance, we shall discuss these points under the assumption that there are no such correlations. The first concerns the fact that the tunneling matrix elements are randomly distributed following a Gaussian law, instead of being constant as assumed in the above discussion. However, it can easily be checked that, in the Poissonian case, this simply amounts to performing first the integral over the tunneling matrix elements distribution. The second point concerns the need to use the regularized form Eq. (3.1) instead of its nondegenerate approximation Eq. (3.2). Let us now see the effect of using this more correct formula which takes quasidegeneracies fully into account by treating the corresponding 2×2 matrix exactly. In this case the relevant function of ξ_i is equal to

$$y_i = \xi_i \left((C_N)^2 - \sqrt{(C_N)^4 + \left(\frac{2v_i}{\xi_i} \right)^2} \right) \quad ; \quad C_N = \frac{D^2 N^2}{v_t^2} \quad (3.13)$$

which is equal to ξ_i^{-1} for $\xi_i \gg C_N^{-1}$ but saturates to a value of $1/v_t$ for smaller values of ξ_i . This implies that $\hat{p}_0(q)$ has a singularity of the type described above only for q less than $v_t/(ND)$. This in turn involves a departure of $\hat{p}(q)$ from pure exponential behavior when q becomes of the order of v_t/D , and hence for normalized splittings of the order of D/v_t , which in unnormalized units correspond to splittings of order v_t . We recover in this way the intuitive picture discussed above, namely that the basic role of the regularization is to prevent having splittings of size larger than the root mean square deviation of the v_i , whereas the distribution for smaller values remains unaffected. For the distribution $p(x)$ displayed in Fig. 2, the effect of this regularization cannot be observed because it only affects the range $x \geq v_t^{-1} = 10^4$ which is not covered in this linear scale. It is however clearly seen in Fig. 1, as well as in the Figs. 3 and 4 of the following section.

IV. THE CASE OF EFFICIENT BARRIERS

Let us now consider the more complicated case of systems for which transport limitation induces strong correlation between the symmetry classes. As we have emphasised in the previous section, the correlation between chaotic states inside a symmetry class has little or no influence on the distribution of the shifts, δ^+ and δ^- , of the regular level, due to

their coupling with the chaotic states. Therefore, even in the case where there exist efficient barriers to transport in the chaotic region, the shifts δ_{\pm} should be still distributed according to the Cauchy distribution derived in the preceding section. In fact, the main effect of such barriers is to induce strong correlations between the E_i^+ and the E_i^- , which only affect the distribution of the splittings themselves $\delta = |\delta^+ - \delta^-|$. Another consequence of the presence of partial barriers which may also influence the distribution of splittings is that it may yield some inhomogeneity of the variance of the tunneling matrix elements, as well as of the correlation between chaotic states. We shall come back to this point at the end of the section when comparing our findings with exact results calculated numerically using Monte-Carlo techniques.

A. Derivation of Eqs. (2.8)-(2.10)

To lighten somewhat the notation we use in this subsection scaled energies $e_i^{\pm} = E_i^{\pm}/D$ (for which the mean density of states around the regular level is therefore one), and note $\bar{v}_t = v_t/D$ and $\bar{\alpha} = \alpha/D$. We use the following modeling of our problem. First, we shall ignore any correlation between chaotic states inside each symmetry class and merely require that the mean density of chaotic states be equal to D around $E_R = 0$. To normalize the number of states to N , we choose the variables e_i^+ and e_i^- both distributed according to a Gaussian law, which we take to be $N^{-1}e^{-\pi(e_i^{\pm}/N)^2}$. In this way, the density at zero is $1/N$ for each level (and thus the total density is 1). Since we shall consider the $N \rightarrow \infty$ limit, this can be thought of as a flat distribution on the scale of a mean level spacing, the Gaussian form being just introduced to normalize in a proper way the distribution. Again, we take the chaotic states to be identically distributed random variables, the e_i^{\pm} are independent of each other for different values of i . The correlations between the two parity sequences of chaotic states must, however, be implemented, which we shall do assuming that the $(e_i^+ - e_i^-)$ have a Gaussian distribution characterized by its width $\bar{\alpha}$. As discussed in section II $\bar{\alpha}$ is related to a characteristic time necessary for a classical trajectory to travel from one regular island to its symmetric counterpart. To justify this construction, let us consider for instance the ensemble of Eq. (2.5) introduced in section II. In this case, it was seen that the Hamiltonians of the two symmetry sectors are related to one another by adding a GOE matrix, the off-diagonal elements of which have variance $(2\sigma)^2$, related through Eqs. (2.6)-(2.7) to the classical flux Φ crossing the partial barrier. As is well-known, the resulting spectrum is formally the same as the result of letting the levels moving according to an interacting Brownian motion during a time $\Lambda = \sigma^2/D^2$, as described by Dyson [24]. However, for short times (i.e. small Λ), it is generally accepted that an interacting diffusion process can be replaced by a free one [25], which here means that the $(E_i^+ - E_i^-)$ follow a Gaussian distribution, of width $\alpha = \sqrt{2}(2\sigma)$. This also follows more specifically from the results shown in [26]. There it was shown that a randomly distributed sequence diffusing over a short time Λ only acquires correlations on a scale of $\sqrt{\Lambda}$. (Note that for $\alpha^2 \gg D^2$, this modeling becomes essentially meaningless. For instance it is not possible anymore to specify unambiguously what $E_{i'}^-$ is to be associated with a given E_i^+ when considering the variance of their difference. This is not of great importance, though, since the final result shows that in this case the truncated Cauchy behavior described in section III is recovered anyhow.) Thus the joint probability

distribution function of the e_i^+ and the e_i^- is

$$P(e_i^+, e_i^-) = \left(\frac{1}{\sqrt{2\pi\bar{\alpha}N}} \right)^N \prod_{i=1}^N \exp \left(-\pi \left(\frac{e_i^+}{N} \right)^2 + \frac{1}{2\bar{\alpha}^2} (e_i^+ - e_i^-)^2 \right). \quad (4.1)$$

Note the asymmetric treatment of e_i^+ and e_i^- . This simply means that, because of the very strong correlation between the two sets of eigenvalues, the large-scale distribution of one set entirely determines that of the other.

For more structured ensembles than the one of Eq. (2.5), the actual diffusion process, while “turning on” the transport parameters Λ_n ’s from zero to their actual value, may be noticeably more complicated. Nevertheless, because the final splitting results from the average effect of the coupling with a large number of chaotic states, it is natural to assume that a kind of central limit theorem is involved and that the form Eq. (4.1) can also be used in practice (we shall discuss this question in greater detail in the next subsection). As mentioned in section II however, there will not be anymore a simple relationship between α and the transport parameters of these ensembles.

Now the problem is to compute the splitting distribution. We shall disregard in the following the complications created by the inclusion of the complete expression Eq. (3.1) for the shifts, since this only causes a cutoff at values of δ equal to v_t , as was already discussed in the previous Section. As a further simplification we shall take for a moment the tunneling matrix elements as being constant (equal to v_t) and shall come back later to the (slight) modifications to the result due to averaging over their Gaussian distribution. Introducing the scaled variable $X = (\delta^+ - \delta^-)/D$, ($\delta/D = |X|$) the distribution of X is obtained as

$$p(X) = \int \prod_{i=1}^N de_i^+ de_i^- P(e_i^+, e_i^-) \delta \left(X - \bar{v}_t^2 \sum_{i=1}^N \left(\frac{1}{e_i^+} - \frac{1}{e_i^-} \right) \right). \quad (4.2)$$

Again, this integration can be factorized by introducing the Fourier transform $F(q)$ of $p(X)$, and everything can be reduced to quadratures. The details are a trifle tedious and are therefore relegated to Appendix A. The final result is

$$p(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(q) e^{-iqx} dx \quad (4.3)$$

$$F(q) = \exp \left(-\frac{\bar{\alpha}}{\sqrt{2\pi}} \Psi(\tilde{q}) \right) \quad ; \quad \tilde{q} = \frac{\bar{v}_t^2}{\bar{\alpha}} q. \quad (4.4)$$

Here $\Psi(\tilde{q})$ is given by the expression

$$\Psi(\tilde{q}) = \int_{-\infty}^{\infty} dy \Phi \left(\frac{\sqrt{8}\tilde{q}}{|1-y^2|} \right), \quad (4.5)$$

$$\Phi(z) = 2 \int_0^{\infty} \frac{dt}{t^3} (1 - \cos zt) e^{-1/t^2}. \quad (4.6)$$

For ease of reference we give the following integrals, which are derived for completeness in Appendix B:

$$\int_{-\infty}^{\infty} \Phi(1/y) dy = \pi^{3/2} \quad (4.7)$$

$$\int_{-\infty}^{\infty} \Phi(1/y^2) dy = \sqrt{2\pi} \Gamma(3/4) \quad (4.8)$$

An asymptotic study of the function $\Phi(z)$ for $z \gg 1$ and $z \ll 1$ yields

$$\begin{aligned} \Phi(z) &= -z^2 \ln z + O(z^2) & (z \ll 1) \\ &= +1 + O(z^{-1}) & (z \gg 1), \end{aligned} \quad (4.9)$$

(the prefactors given here are actually correct, but we shall not need them, and the order of magnitude is easy to obtain). Therefore, $\Psi(\tilde{q})$ basically gives a measure of the domain of y such that $\tilde{q}/(1-y^2)$ is larger than one. For $\tilde{q} \gg 1$, this is obviously of order $\sqrt{\tilde{q}}$. It can moreover be evaluated more precisely by noting that in this range of \tilde{q} , $\Phi(\sqrt{8\tilde{q}}/|1-y^2|) \simeq \Phi(\sqrt{8\tilde{q}}/y^2)$. This is the case, because in the range of y where $|1-y^2|^{-1} \neq y^{-2}$, i.e. for y not large, both $\tilde{q}/|1-y^2|$ and \tilde{q}/y^2 are much larger than one if \tilde{q} is, and therefore Φ saturates to its asymptotic value 1 anyhow. One therefore finds for $\tilde{q} \gg 1$

$$\Psi(\tilde{q}) \approx \int_{-\infty}^{\infty} dy \Phi\left(\frac{\sqrt{8\tilde{q}}}{y^2}\right) = 2^{3/4} \Gamma(3/4) \sqrt{2\pi|\tilde{q}|}. \quad (4.10)$$

and therefore

$$F(q) \approx F_{\infty}(q) = \exp\left(-2^{3/4} \Gamma(3/4) \sqrt{\bar{\alpha} \bar{v}_t^2 |q|}\right) \quad \text{for } q \gg \frac{\bar{\alpha}}{\bar{v}_t^2}. \quad (4.11)$$

For $\tilde{q} \ll 1$ on the other hand, $\tilde{q}/(1-y^2)$ is large only in the neighborhood of $y = 1$. From this follows that one can restrict oneself to the range of integration $y \sim \pm 1$. One obtains

$$\begin{aligned} \Psi(\tilde{q}) &\approx 2 \int_{1-\tilde{q}}^{1+\tilde{q}} dy \Phi\left(\frac{\sqrt{8\tilde{q}}}{(1-y)(1+y)}\right) \\ &\approx 2 \int_{-\infty}^{\infty} dt \Phi\left(\frac{\sqrt{2\tilde{q}}}{t}\right) = (2\pi)^{3/2} \tilde{q}, \end{aligned}$$

and therefore

$$F(q) \approx F_0(q) = \exp\left(-2\pi \bar{v}_t^2 |q|\right) \quad \text{for } q \ll \frac{\bar{\alpha}}{\bar{v}_t^2}. \quad (4.12)$$

It remains to perform the inverse Fourier transform Eq. (4.4), and to deduce the asymptotic behavior of $p(X)$ from the one of $F(q)$. For large X , $p(X)$ is dominated by the singularities of its Fourier transform, which here means the derivative discontinuity at the origin. Therefore for $\bar{v}_t > X \gg \bar{v}_t^2/\bar{\alpha}$ one can use in Eq. (4.4) the asymptotic $q \ll \bar{\alpha}/\bar{v}_t^2$ approximation $F_0(q)$ of $F(q)$. Applying the inverse Fourier transformation yields an almost perfect Cauchy distribution of the form

$$p(X) = \frac{2\bar{v}_t}{X^2 + 4\pi^2 \bar{v}_t^2}. \quad (4.13)$$

The above result amounts to adding independently the variable δ^+ and δ^- , distributed as given by Eq. (3.6), and to neglect the correlations between the two symmetry classes. As mentioned in [9], this is indeed quite natural since splittings larger than $v_t^2/\bar{\alpha}$ are due to chaotic levels coming closer than a distance $\bar{\alpha}$ of the regular level. Since $\bar{\alpha}$ can be viewed as the scale on which chaotic levels are correlated, chaotic states contributing to $p(X)$ for $X \gg \bar{v}_t^2/\bar{\alpha}$ can therefore be considered as essentially decorrelated from their symmetric counterparts. In the language of section 2, the splittings we are looking at here are so large that the tunneling is always mediated by a single state.

Let consider now the range $X \ll \bar{v}_t^2/\bar{\alpha}$, for which the splitting distribution is affected by correlation between symmetry classes. In that case, the term $\exp(-iqX)$ in the inverse Fourier transformation Eq. (4.3) is essentially constant in all the range $q \leq \alpha/v_t^2$ where $F(q)$ differs from its asymptotic behavior $F_\infty(q)$. Noting that

$$\frac{1}{2\pi} \left(\int_{-\infty}^{\infty} F_\infty(q) \exp(-iqx) dq \right) = \lambda^{-1} G(X/\lambda) \quad ; \quad \lambda = \sqrt{8}\Gamma^2(3/4)\bar{\alpha}\bar{v}_t^2, \quad (4.14)$$

where $G(x)$ is the inverse Fourier transform of $\exp(-\sqrt{q})$ as defined in Eq. (2.11). One therefore has

$$p(X) = \lambda^{-1} G(X/\lambda) + K \quad (4.15)$$

where K is the constant

$$K \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dq (F(q) - F_\infty(q)) \quad (4.16)$$

For small $\bar{\alpha}$ however, K is of order $\bar{\alpha}^2/\bar{v}_t^2$, when $\lambda^{-1}G(X/\lambda)$ range from order $1/(\bar{\alpha}\bar{v}_t^2)$ at $X = 0$ to $\bar{\alpha}^2/\bar{v}_t^2$ at its lowest value, i.e. at the crossover $X \sim \bar{v}_t^2/\bar{\alpha}$ between the Cauchy-like and G-like behavior. Therefore it can usually be neglected, although in some special circumstances it shows up as a small plateau between these two regimes; we shall disregard it from now on. Then, the large- X behavior of $\lambda^{-1}G(X/\lambda)$ is dominated by the \sqrt{q} singularity at the origin of $F_\infty(q)$, so that it goes as $X^{-3/2}$ as $X \rightarrow \infty$. This is in fact the hallmark of the $X \ll \bar{v}_t^2/\bar{\alpha}$ regime we are discussing.

Finally, one has to take into account the fact that the tunneling matrix elements are not constant, but randomly distributed. As can be seen in the derivation of Eq. (4.4) (see the remark below Eq. (A4)) this merely amounts to replacing the expression for $\Psi(q)$ given in eq. (4.5) by the function $\bar{\Psi}(q)$ defined as follows:

$$\bar{\Psi}(q) = \left\langle \Psi \left(v^2 q / \alpha \right) \right\rangle_v, \quad (4.17)$$

where the brackets denote averaging over the v 's, which we take to have a Gaussian distribution with variance v_t^2 . This new function is of course much more complicated than the original one, but its asymptotic behavior for small or large values of q is readily obtained. Indeed, for $q \ll 1$, $\Psi(q)$ is proportional to $|q|$, so that the average is obtained by replacing v by v_t . On the other hand, for $q \gg 1$, $\Psi(q)$ is proportional to $\sqrt{|q|}$, so that its average over a Gaussian distribution is obtained by replacing v by $\sqrt{2/\pi}v_t$. Using these facts together with the above estimates for the behavior of $p(X)$ one finally obtains the result stated in

Eqs. (2.8–2.10) in section II. Some trivial differences: There we consider splittings as being always positive, whereas in the above computation we treated positive and negative splittings separately: this introduces a new factor of two. Further, we have made the dependence on D explicit, which in particular means replacing $p(X)$ by $Dp(X/D)$ as well as replacing $\bar{\alpha}$ and \bar{v}_t by their original expressions.

B. More Structured Ensembles

For the simple ensemble Eq. (2.5), the two main assumptions we made to replace the exact distribution by Eq. (4.2), namely to neglect correlation of chaotic states among a given symmetry class and to replace the interacting brownian diffusion by a free one for small $\bar{\alpha}$ are really under control. Indeed, section III gives full justification of the first assumption, and the second can be seen as a simple consequence of standard perturbation theory. And actually, as shown in Fig. 3, one can see that our analytical findings perfectly agree with an “exact” Monte-Carlo evaluation of the splitting distribution generated by the ensemble Eq. (2.5). We stress that in this very simple case the parameter α that we are using is (for small α) simply related to the variance σ^2 of the non-diagonal matrix elements of $(GOE)_A$ (indeed $\alpha^2 = 2(2\sigma)^2$), and that therefore there are no adjustable parameters in this comparison.

More structured ensembles deserve however some further discussion. Let consider for instance the ensemble relevant to the quartic oscillator system used as illustration in section II. Symbolically, this ensemble can be written as [9]

$$H_{\text{qo}}^{\pm} = \begin{pmatrix} E_R & \{v\} & 0 & 0 \\ \{v\} & (GOE)_1 & (GOE)^{\pm}(\Lambda_{12}) & (GOE)^{\pm}(\Lambda_{13}) \\ 0 & (GOE)^{\pm}(\Lambda_{12}) & (GOE)_2^{\pm} & 0 \\ 0 & (GOE)^{\pm}(\Lambda_{13}) & 0 & (GOE)_3^{\pm} \end{pmatrix}, \quad (4.18)$$

where the subscript \pm again indicates ensemble which are independent in the $+$ and $-$ symmetry class. Noting D_{tot} the total density of states (in a given symmetry class), $(GOE)_i$ stands for a Gaussian Orthogonal Ensemble such that the mean level density in the center of the semicircle is $f_i D_{\text{tot}}$, and $(GOE)^{\pm}(\Lambda_{jk})$ represent Gaussian distributed independent matrix elements of variance $\sigma_{jk}^2 = \Lambda_{jk} D_{\text{tot}}^2$. (For the configuration of the quartic oscillators corresponding to Fig. 1, one has $f_1 = 0.5$, $f_2 = 0.2$, $f_3 = 0.3$, and $\Lambda_{12} = 0.14$, $\Lambda_{13} = 0.11$.) For such complicated ensembles the two assumptions concerning the irrelevance of intra-class correlations and essentially Gaussian distribution of the $(E_i^+ - E_i^-)$ are presumably equally well fulfilled here as in the simple case of Eq. (2.5). What is lost however is the uniformity of the distribution of the tunneling matrix elements and of the variance of the $(E_i^+ - E_i^-)$. Indeed, in the above example, a diagonalization of the chaotic part of the Hamiltonian is going to transfer some tunneling matrix elements from the block connecting E_R to $(GOE)_1$ to the ones connecting $(GOE)_2$ and $(GOE)_3$. One may end in this way with three different scales for the variance of the tunneling matrix elements as well as for the parameter α (one for each (GOE) block).

More generally, the typical situation will be that a [possibly large] number of (GOE) blocks, $(GOE)_1$, $(GOE)_2$, ..., $(GOE)_K$ are involved in the tunneling process. After diagonalization of the chaotic part of the Hamiltonian, both the variance of the tunneling matrix

elements, and the degree of correlation between symmetry classes, will be block dependent. Each block $(GOE)_k$ ($k = 1, \dots, K$) would have then to be characterized by a tunneling parameter v_k and a transport parameter α_k (α_k and v_k highly correlated), in addition to its dimension $N_k = f_k N$. Let us introduce the notation

$$I(\alpha, v_t; q) = -\frac{\alpha}{D\sqrt{2\pi}}\Psi(v_t^2 q/\alpha D) , \quad (4.19)$$

where $\Psi(\tilde{q})$ is defined by Eqs. (4.4). A straightforward modification⁴ of the derivation of Eq. (4.4) gives that taking into account the block dependence of α_k and v_k merely amounts to replacing this equation (i.e. $F(q) = \exp(-I(\alpha, v_t; q))$) by

$$F(q) = \exp\left(-\sum_{k=1}^K f_k I(\alpha_k, v_k; q)\right) . \quad (4.20)$$

Inspection of Eqs. (4.11) and (4.12) then shows that they remain valid provided α and v_t are defined now as

$$v_t^2 \equiv \sum_k f_k v_k^2 \quad (4.21)$$

$$\alpha \equiv \frac{1}{v_t^2} \left(\sum_k f_k \alpha_k^{1/2} v_k \right)^2 . \quad (4.22)$$

Multiplying Eq. (4.21) by N , Nv_t^2 appears as the [average] square norm of the projection of the quasimode Ψ_R^\pm on the chaotic space. It is therefore independent of the chaotic phase space structure. This, for instance, allows to compute v_t from the variance of the tunneling matrix elements *before* diagonalization of the chaotic part of the Hamiltonian. The parameter α and v_t have moreover a certain number of intuitively clear properties: If all v_k are multiplied by a constant factor, the effective tunneling element v_t is multiplied by the same factor, whereas α is unaffected. Further, if all v_k are identical, then the effective tunneling element is the same. On the other hand, the same is not true of α : If all α_k and all v_k are taken to be equal, the effective efficiency of the classical barrier now depends on the number of different components of phase space through which tunneling can take place. Further, we see that any components with negligible values of v_k will contribute negligibly both to v_t and α . Thus we can identify a given part of phase space through which tunneling actually occurs and limit ourselves to it.

With the definitions Eqs. (4.21) and (4.22) of v_t and α , structured ensemble are therefore seen to behave in essentially the same way as the simple ensemble Eq. (2.5). The, only difference is that the condition of validity of Eqs. (4.11) and (4.12), that is respectively

⁴In Eq. (A4), $\lim_{N \rightarrow \infty} (1 - I(q)/N)^N$ has to be replaced by

$$\lim_{N \rightarrow \infty} \left[\prod_{k=1}^K (1 - I(\alpha_k, v_k; q)/N)^{f_k N} \right] .$$

$$q \gg (\alpha_k D)/v_k^2 \quad (4.23)$$

$$\text{and} \quad v_k^{-1} \ll q \ll (\alpha_k D)/v_k^2, \quad (4.24)$$

must now be fulfilled for all k 's. The transition between the different regimes of the distributions may therefore be less sharp than for the ensemble Eq. (2.5).

If the partial barriers structures were to become highly developed, say to the point that the ensemble could meaningfully be described in terms of band matrices, then obviously the issue of localization should have to be considered. In this case, the orders of magnitude of the α_k might become comparable to those of the v_k , and most of the splitting distribution might be actually in a transition-like regime. This is exactly the sort of problems we pointed out in our earlier discussion of the physical situation. However, as long as the α_k 's are clearly larger than the v_k 's, the transition from one regime to another should still take place on a short scale as compared to the range spanned by the distribution. Physically speaking, this condition amounts to saying that classically forbidden processes are always much slower than classically allowed ones. In that case, the form of the result should not be noticeably affected. For example, as seen in Fig 4, the distribution resulting from the ensemble described in Eq. (4.18) still perfectly follows the predicted form Eqs. (2.8)-(2.10) (note however that α is now a tunable parameter).

V. CONCLUSION

As a conclusion, we have provided in this paper an analytical study of the splitting distributions generated by ensembles of random matrices constructed in [9] to model a tunneling process in the chaos assisted regime. The original ensembles may contain such a complicated structure that a general answer to this problem may seem a priori out of reach. Nevertheless, it turns out that only the average size of the tunneling matrix elements and the degree of correlation between the chaotic spectra in the different symmetry class affect the distribution, and that therefore the problem can be reduced to a simpler formulation which is tractable.

The basic reason for the considerable simplifications encountered was in essence already pointed out in [9]. It is due to the fact that for large splittings only the situation of near-resonance to a given state of the chaotic sea is of relevance. To this obvious remark, we only need add that for the case in which efficient barriers are at work, the tunneling operates not through single states, but through quasi-degenerate doublets of states of opposite parity. These are of course less efficient in promoting tunneling, since the particle requires a time of the order of the width of the doublet to reach the symmetrical torus. In either case, the behavior is determined by rather natural probabilistic considerations. It turns out to be sufficient to consider only the probability of one single eigenvalue being near the tunneling state, so that correlations between eigenvalues and the like could be safely ignored. Further, the very simplicity of the physical picture given here results in it being fairly robust to changes in minor details of the model. Thus it does not appear necessary that all states in the chaotic sea should participate equally in the tunneling process, nor that the couplings should be uniform. In fact, the main limitations of our result seem to be the ones related to localization phenomena. If the structure of the barriers in phase space is sufficiently

complicated, it is possible that localization effects, associated to the presence of a large number of partial barriers, become as effective in limiting tunneling from one quasimode to its symmetric partner as the initial classically forbidden process. In that case, the splitting distributions we have obtained would not be relevant anymore. However, this should not be a too severe limitation, and it should generally be possible to determine for any given system whether this takes place or not. When it does not, the picture of tunneling in the presence as well as in the absence of barriers to transport is indeed the one we gave. This is substantiated by the numerical work done: In particular, we showed that not only the simplest model of a barrier gives results in good agreement with theoretical predictions, but also a highly specific random matrix ensemble constructed explicitly in order to model chaos-assisted tunneling in a system of coupled quartic oscillators was well fitted by the theoretical predictions, as were also the actual splitting distribution for this system.

This might possibly open up a way to identify chaos-assisted tunneling in experimental systems. In such systems, the exhaustive study of the classical mechanics necessary to produce a satisfactory random matrix ensemble would probably not be feasible. Nevertheless the above remarks strongly suggest that if chaos-assisted tunneling is present, the splitting distribution will reflect the fact by showing a highly specific and well-characterized behavior. Indeed, as discussed throughout this paper, only the scale of the distribution and the position of the transitions between the different regimes are system dependent, but the shape of the distribution is essentially universal. In particular the experimental detection of a transition from a $\delta^{-3/2}$ behavior, characteristic of the G -like regime, to a δ^{-2} behavior characteristic of the Cauchy-like regime, would be a powerful argument in favor of the presence of chaos-assisted tunneling.

ACKNOWLEDGMENTS

We acknowledge many helpful discussions with Oriol Bohigas, as well as with Steven Tomsovic which we thank in addition for providing us with the numerical data of the quartic oscillators system used in Fig. 1. Denis Ullmo wants to thank the Institute for Nuclear Theory, Seattle WA, where parts of this work has been done. François Leyvraz would like to thank the Division de Physique Théorique for its kind hospitality during his sabbatical stay there where most of this work was done, as well as acknowledge financial support by DGAPA during that same period. The Division de Physique Théorique is “Unité de Recherche des Universités Paris 11 et Paris 6 Associée au CNRS”.

APPENDIX A: COMPUTATION OF THE DISTRIBUTION FUNCTION

Denote by brackets the integration over e_i^+ and e_i^- with the weight function $P(e_i^+, e_i^-)$ (see Eq (4.1)). We define

$$p(X) = \left\langle \delta \left(x - \bar{v}_t^2 \sum_{i=1}^N \left(\frac{1}{e_i^+} - \frac{1}{e_i^-} \right) \right) \right\rangle, \quad (\text{A1})$$

$$F(q) = \int_{-\infty}^{\infty} F(X) e^{iqX} dx$$

$$= \left\langle \exp \left(iq \bar{v}_t^2 \sum_{i=1}^N \left(\frac{1}{e_i^+} - \frac{1}{e_i^-} \right) \right) \right\rangle \quad (\text{A2})$$

This last expression factorizes in N factors, each of which is a double integral. Denoting the corresponding average over e^+ and e^- also by brackets, one obtains:

$$\begin{aligned} F(q) &= \left\langle \exp \left(iq \bar{v}_t^2 (1/e^+ - 1/e^-) \right) \right\rangle^N \\ &= \left(1 - \left\langle 1 - \exp \left(iq \bar{v}_t^2 (1/e^+ - 1/e^-) \right) \right\rangle \right)^N \\ &= \left(1 - \frac{I_N(q)}{N} \right)^N \end{aligned} \quad (\text{A3})$$

where the last line defines $I_N(q)$. The reason for this manipulation is that in this way I_N goes to a finite limit I as $N \rightarrow \infty$ and therefore

$$\lim_{N \rightarrow \infty} F(q) = \lim_{N \rightarrow \infty} (1 - I(q)/N)^N = \exp(-I(q)) . \quad (\text{A4})$$

Note moreover that taking into account the fact that the tunneling matrix elements are random variable of variance v_t^2 instead of being constant just amount to understand $\langle \cdot \rangle$ as containing a further integral on the tunneling matrix elements distribution. This introduce no further difficulties in the calculation of I , except for still heavier notations. We shall therefore not consider it in this appendix, and just modify the final result in the appropriate way at the end of section IV A.

One finds

$$I(q) = \frac{1}{\sqrt{2\pi\bar{\alpha}}} \int de^+ de^- \left(1 - \exp \left(i \bar{v}_t^2 q (1/e^+ - 1/e^-) \right) \right) \exp \left(-\frac{(e^+ - e^-)^2}{2\bar{\alpha}^2} \right) , \quad (\text{A5})$$

since, the above integral being convergent, $\lim_{N \rightarrow \infty} I_N$ is just obtained by dropping the term $-\pi(e^+/N)^2$ in the exponent of $P(e^+, e^-)$ (see Eq. (4.1)). Making the successive transformations $y = (1/e^+ + 1/e^-)/(1/e^+ - 1/e^-)$, $w = (1/e^+ - 1/e^-)$, followed by $t = w(\bar{\alpha}(1 - u^2)/\sqrt{8})$, $I(q)$ can be expressed as

$$\begin{aligned} I(q) &= \frac{8}{\sqrt{2\pi\bar{\alpha}}} \int \frac{dy}{(1 - y^2)^2} \int \frac{dw}{|w|^3} (e^{i w \bar{v}_t^2 q} - 1) \exp \left(-\frac{8}{\bar{\alpha}^2(1 - y^2)^2 w^2} \right) \\ &= \frac{\bar{\alpha}}{\sqrt{2\pi}} \int dy \int \frac{dt}{|t|^3} \left(1 - \cos \left(\frac{\sqrt{8} q \bar{v}_t^2}{\bar{\alpha}(1 - y^2)} \right) \right) e^{-1/t^2} . \end{aligned} \quad (\text{A6})$$

If we now introduce $\Phi(z)$ as in the text,

$$\Phi(z) = 2 \int_0^\infty \frac{dt}{t^3} (1 - \cos zt) e^{-1/t^2} , \quad (\text{A7})$$

one easily obtains

$$I(q) = \frac{\bar{\alpha}}{\sqrt{2\pi}} \int dy \Phi \left(\frac{\sqrt{8} q \bar{v}_t^2}{\bar{\alpha}|1 - y^2|} \right) . \quad (\text{A8})$$

From this follows the formula given in the text:

$$F(q) = \exp \left(-\frac{\bar{\alpha}}{\sqrt{2\pi}} \Psi(\bar{v}_t^2 q / \bar{\alpha}) \right), \quad (\text{A9})$$

where $\Psi(\tilde{q})$ is given by the expression

$$\Psi(\tilde{q}) = \int_{-\infty}^{\infty} dy \Phi \left(\frac{\sqrt{8}\tilde{q}}{|1-y^2|} \right). \quad (\text{A10})$$

APPENDIX B: SOME USEFUL INTEGRALS

We first give another expression for $\Phi(y)$:

$$\Phi(y) = \int_0^{\infty} dt (1 - \cos \frac{y}{\sqrt{t}}) e^{-t}, \quad (\text{B1})$$

which is obtained from the original definition by substituting $1/t^2$ by t . From this follows

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(1/y) dy &= \int_{-\infty}^{\infty} \Phi(y) \frac{dy}{y^2} \\ &= \int_0^{\infty} dt e^{-t} \int_{-\infty}^{\infty} (1 - \cos \frac{y}{\sqrt{t}}) \frac{dy}{y^2} \\ &= \int_0^{\infty} \frac{dt}{\sqrt{t}} e^{-t} \int_{-\infty}^{\infty} \frac{1 - \cos y}{y^2} dy \\ &= \Gamma(1/2) \int_{-\infty}^{\infty} \frac{\sin y}{y} dy = \pi^{3/2} \end{aligned} \quad (\text{B2})$$

The other integral is handled similarly:

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(1/y^2) dy &= \int_{-\infty}^{\infty} \Phi(y^2) \frac{dy}{y^2} \\ &= \int_0^{\infty} \frac{dt}{t^{1/4}} e^{-t} \int_{-\infty}^{\infty} \frac{1 - \cos y^2}{y^2} dy \\ &= 2\Gamma(3/4) \int_{-\infty}^{\infty} \sin y^2 dy = \sqrt{2\pi} \Gamma(3/4). \end{aligned} \quad (\text{B3})$$

REFERENCES

- [1] A.M. Ozorio de Almeida, *Hamiltonian system: Chaos and quantization*, Cambridge University Press, 1988.
- [2] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, (Springer-Verlag, Berlin, 1990).
- [3] M.-J. Giannoni, A. Voros, and J. Zinn-Justin, eds., *Chaos et Physique Quantique / Chaos and Quantum Physics*, Proceeding of les Houches summer school, session LII 1989. North-Holland, Amsterdam, 1991.
- [4] W.A. Lin and L.E. Ballentine, Phys. Rev. Lett. **65** (1990) 2927; Phys. Rev. A **45** (1992) 3637.
- [5] F. Grossmann, T. Dittrich, P. Jung, and P. Hänggi, Phys. Rev. Lett. **67** (1991) 516; Z. Phys. B **84** (1991) 315; J. Stat. Phys., **70** (1993) 229; R. Utterman, T. Dittrich, and P. Hänggi, *Tunneling and the Onset of Chaos in a Driven Bistable System*, (preprint 1993).
- [6] J. Plata and J.M. Gomez Llorente, J. Phys. A **25** (1992) L303.
- [7] O. Bohigas, S. Tomsovic, D. Ullmo, Phys. Rep. **223**, (1993) 43.
- [8] O. Bohigas, D. Boosé, R. Egydio de Carvalho, and V. Marvulle, Nucl. Phys. **A560** (1993) 197.
- [9] S. Tomsovic and D. Ullmo, Phys. Rev. E **50** (1994) 145.
- [10] M.J. Davis and E.J. Heller, J. Chem. Phys. **75** (1981) 246.
- [11] See e.g. appendices 10 and 11 of : V.I. Arnold, *Mathematical methods of classical mechanics*, Springer Verlag, 1978.
- [12] V.P. Maslov and M.V. Fedoriuk, *Semiclassical approximation in quantum mechanics*, (Reidel, Dordrecht, 1981) [original Russian edition 1965].
- [13] M. Wilkinson, Physica **21D** (1986) 341; J. Phys. **A20** (1987) 635.
- [14] S.C. Creagh, J. Phys. A **27** (1994) 4369.
- [15] R. Roncaglia, L. Bonci, F.M. Izrailev, B.J. West, and P. Grigolini, Phys. Rev. Lett. **73** (1994) 802.
- [16] C.E. Porter, editor, *Statistical properties of spectra: Fluctuations*, Academic Press, New York, 1965.
- [17] M.L. Mehta, *Random Matrices and the statistical properties of energy levels*, Academic Press, New York, 1967. New enlarged edition, 1990.
- [18] O. Bohigas, M.-J. Giannoni and C. Schmit, Phys. Rev. Lett. **52** (1984) 1; Journal Physique Letters **45** (1984) L1015.
- [19] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2 (Wiley, 1957).
- [20] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press 1980).
- [21] F. von Oppen, private communication.
- [22] G. López, P.A. Mello, T.H. Seligman, Zeitschr. f. Physik A **302** (1981) 351.
- [23] P.A. Mello in *Mesoscopic Quantum Physics* Les Houches 1994.
- [24] F.J. Dyson, J. Math. Phys. **3** (1962) 1191.
- [25] for Coulombic systems see e.g. M. Medina-Noyola, Phys. Rev. Lett. **60** (1988) 2705; M. Medina-Noyola and A. Vizcarra-Rendón, Phys. Rev. A **32** (1985) 3596.
- [26] F. Leyvraz and T.H. Seligman, J. Phys. A **23** (1990) 1555.

FIGURES

FIG. 1. Comparison between the quartic oscillator's tunneling splitting distribution (square symbols) and the predicted form Eqs.(2.8)-(2.10) for two different groups of tunneling tori. Except for their presentation, the quartic oscillator's data are the same as those in Fig. 13 of Ref. [9]. The transition from the G -like behavior (solid line) to Cauchy-like behavior (chained dot) is clearly seen, in spite of this latter being valid on a much shorter range. (a) Group T_0 (using the notations of Ref. [9]), with $v_t = 1.1 \cdot 10^{-2}$. (b) Group T_1 , with $v_t = 2.5 \cdot 10^{-2}$. The transport parameter has the same value $\alpha = .04$ in both cases, consistent with the fact that the partial barriers structure is the same in both cases. It has been taken into account that only a fraction $D_{\text{eff}} = 0.36D$ of the states are actually participating to the tunneling process.

FIG. 2. Comparison between a Monte Carlo calculated distribution of the reduced variable $x = \delta^\pm(D/v_t^2)$ (solid line) and the Cauchy law Eq. (3.6) (chained dot). The Monte Carlo result is obtained from the numerical diagonalization of 10^5 matrices of size 80×80 , which matrix elements are taken at random with the distribution specified by the ensemble Eq. (2.3) (using $v_t/D = 10^{-4}$). It thus take fully into account the GOE correlations of the chaotic states. Nevertheless, and although a linear scale has been used to emphasize the center of the distribution where the effects of correlations should be the strongest, the two curves are essentially undistinguishable.

FIG. 3. Comparison between a Monte Carlo calculated distribution of splittings δ for the simple ensemble Eq. (2.5) (solid line) and the predicted form Eqs.(2.8)-(2.10). The parameters of the Monte Carlo calculations are $\Lambda = 10^{-2}/8$ (imposing $\alpha/D = 0.1$ for the theoretical curve), $v_t/D = 10^{-4}$, number of matrices: $3 \cdot 10^5$, size of matrices 60×60 . The three regimes: G -like behavior (chained dot), Cauchy-like behavior (dash), and truncation of the Cauchy law for splitting greater than v_t/D are clearly seen.

FIG. 4. Comparison between a Monte Carlo calculated distribution of splittings δ for the ensemble Eq. (4.18) with $f_1 = 0.5$, $f_2 = 0.2$, $f_3 = 0.3$, and $\Lambda_{12} = 0.14$, $\Lambda_{13} = 0.11$ (solid line), and the predicted form Eqs.(2.8)-(2.10). The Monte Carlo calculations has been performed with 10^5 matrices of size 100×100 , using as tunneling parameter $(v_1)^2/D = 10^{-3}$. For the theoretical curves, namely the G -like (chained dot) and Cauchy-like (dash) behaviors, the tunneling parameter is determined by Eq. (4.21) as $(v_t)^2 = (v_1)^2/2$. The transport parameters is however here a tunable parameter, which has been taken equal to $\alpha/D = 0.1$.

Fig. 1b

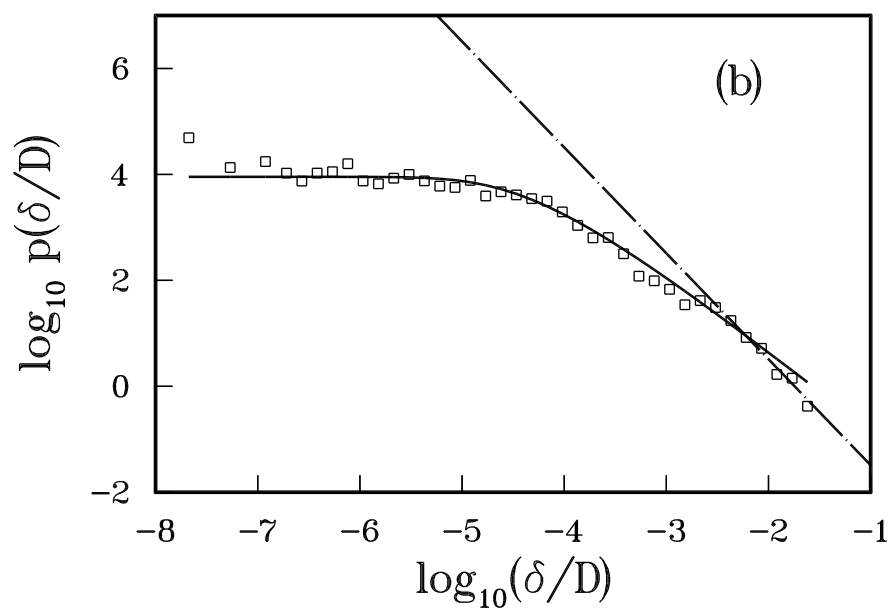


Fig. 2

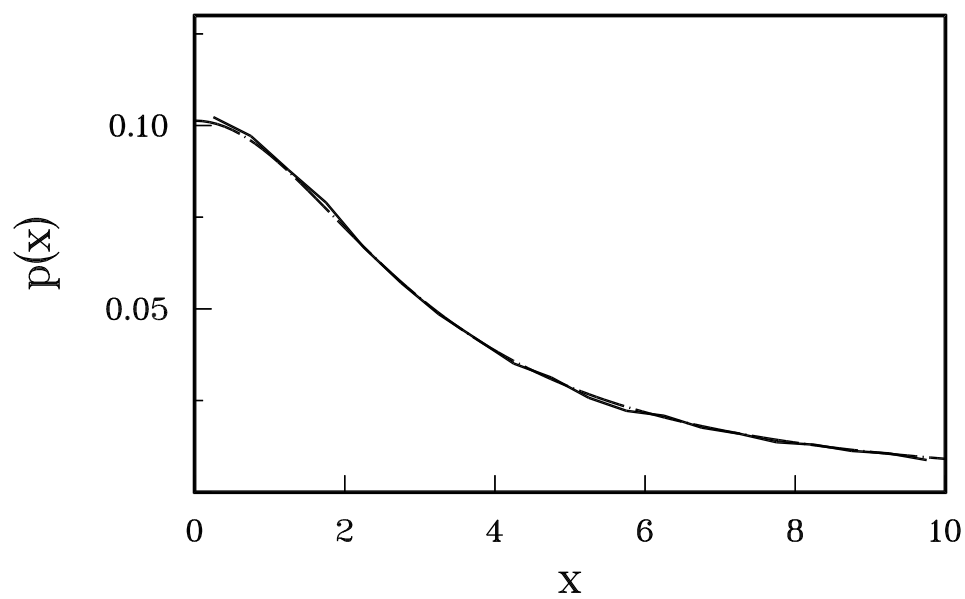


Fig. 3

